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USING EFFECTIVE THEORIES TO STUDY THE CHIRAL PHASE TRANSITION

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RESUMEN
Simetría quiral es la invarianza de la QCD ante rotaciones en el espacio de isospin de los quarks ligeros. La masa no nula de los quarks rompe la simetría pero puede ser restaurada a través de una *transición de fase quiral* que se da a cierta *temperatura crítica*. La transicion puede ser estudiada usando diferentes modelos, entre los cuales se encuentran los modelos σ. En este trabajo se hace un análisis de los modelos σ lineal y no lineal con diferentes aproximaciones. Mostramos que la temperatura crítica es la misma en ambos modelos.

Palabras claves: Simetría quiral, transición de fase quiral, gas de piones.

ABSTRACT
Chiral symmetry is the invariance of the QCD under rotations in the isospin space of the light quarks. The nonzero mass of the quarks break the symmetry but it can be restored through a *chiral phase transition* at a certain *critical temperature*. The transition can be studied through several models, among which are the σ models. An analysis is made on the linear and nonlinear σ models with different approximations. We show that the critical temperature is the same in both cases.

Keywords: Chiral symmetry, quiral phase transition, pion gas.

1. Introduction

A *chiral transformation* is a rotation in the *isospin* space of quarks. When a quark system is invariant under such a transformation, we say that the system is chirally symmetric. This is true in the case of massless quarks, but we know that actually they have mass. So with regard to the light quarks only (*up* and *down*), we can say that the chiral symmetry is an approximated symmetry of the strong interactions [5]. Nonzero mass of the corresponding mesons also breaks the symmetry.

There are two kinds of chiral transformations. The partially conserved Noether current is a vector in one case, and an axial vector in the other case. When working with the up and down quarks, the transformations can be studied through the group $\mathsf{SU}(2)_L \times \mathsf{SU}_R$, which is isomorphic to O(4). This isomorphism is the reason for the Heisenberg magnet model being the unique description of the low energy dynamics of the QCD.

The σ models arise in this context. The σ model has the usual kinetic term and the potential

$$V(\Phi) = \frac{\lambda}{4}\left(\mathbf{\Phi}^2 - f_\pi^2\right)^2, \tag{1}$$

with a field $\mathbf{\Phi}$ having N components, where λ is a positive coupling constant, and f_π is the pion decay constant; this model is renormalizable. In the limit that $\lambda \to \infty$, the potential goes over to a δ-function constraint on the length of the field vector. This is the so-called nonlinear σ model. Both models describe a quark system in which the chiral symmetry is broken, in consistency with the nonzero mass of either the light quarks or the light mesons.

It is interesting to study situations in which a symmetry is restored. The electroweak model has a symmetry-restoring phase transition. If quarks are massless, QCD is also expected to undergo a *chiral phase transition*, which may have implications for high energy nucleus-nucleus collisions. A pion gas with a σ meson undergoes the chiral phase transition. In this work, we use both the σ models to describe it and derive its pressure. The linear σ model is studied through a mean field approximation, while the nonlinear σ model is studied using chiral perturbation theory. The first and second term of the perturbation series are considered. In both studies, the transition is of second-order and the critical temperature is the same, as we will show later.

2. Linear σ model

The linear σ model Lagrangian is

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \mathbf{\Phi}^2\right)^2 - \frac{\lambda}{4}\left(\mathbf{\Phi}^2 - f_\pi^2\right)^2, \tag{2}$$

where λ is a positive coupling constant. The bosonic field $\mathbf{\Phi}$ has N components:

$$\Phi_i(\mathbf{x},t) = \pi_i(\mathbf{x},t), \quad i = 1,\dots,N-1; \qquad \Phi_N(\mathbf{x},t) = v + \sigma(\mathbf{x},t). \tag{3}$$

Let ϕ denote the fields π and σ. In terms of these fields the Lagrangian becomes

$$\mathscr{L} = \tfrac{1}{2}\left(\partial_\mu\phi\right)^2 - \tfrac{\lambda}{4}\left(v^2 - f_\pi^2\right)^2 - \tfrac{\lambda}{2}\left(v^2 - f_\pi^2\right)\left(2v\sigma + \sigma^2 + \pi^2\right) - \tfrac{\lambda}{4}\left(2v\sigma + \sigma^2 + \pi^2\right)^2 \tag{4}$$

At zero temperature, the potential is minimized when $v = f_\pi$. In this particular case, the effective masses are given by $\bar{m}_\pi^2 = \lambda\left(v^2 - f_\pi^2\right) = 0$, $\bar{m}_\sigma^2 = \lambda\left(3v^2 - f_\pi^2\right) = 2\lambda f_\pi^2$, so the Goldstone theorem is satisfied.

Rotating to imaginary time, $\tau = it$, and integrating τ, we calculate the finite-temperature action:

$$S = -\tfrac{\lambda}{4}\left(f_\pi^2 - v^2\right)^2 \beta V + \int \mathrm{d}^4x \Bigg(\tfrac{1}{2}\Big[\left(\partial_\mu\pi\right)^2 - \bar{m}_\pi^2\pi^2 + \left(\partial_\mu\sigma\right)^2 - \bar{m}_\sigma^2\sigma^2\Big] - \lambda v\left(v^2 - f_\pi^2\right)\sigma - \lambda v\sigma\left(\sigma^2 + \pi^2\right) - \tfrac{\lambda}{4}\left(\sigma^2 + \pi^2\right)^2\Bigg), \tag{5}$$

where the abbreviation $\int \mathrm{d}^4x \equiv \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x$ is used. At any temperature v is choosen such that $\langle\sigma\rangle = 0$. This eliminates any one-particle reducible diagrams in perturbation theory, leaving only 1PI diagrams. The simplest approximation at finite temperature is the mean field approximation.

The **generating functional** with source term $\mathbf{J}$ is given by

$$W[\mathbf{J}] = N' \int \mathscr{D}^{N-1}\pi \int \mathscr{D}\sigma \; \exp \int \mathrm{d}^4x \left(\mathscr{L} + J^a\pi^a + j^N\sigma\right) \tag{6}$$

Once the integration over pion fields is done, we have the result

$$W[\mathbf{J}] = N \int \mathscr{D}\sigma \; \exp\Bigg(-\tfrac{i}{2}\int \mathrm{d}^4x' \int \mathrm{d}^4x \; J^a(x')\Delta_{\sigma F}(\mathbf{x}' - \mathbf{x})J_a(x) + \int \mathrm{d}^4x \left(\tfrac{1}{2}\left[\left(\partial_\mu\sigma\right)^2 - \bar{m}_\sigma^2\sigma^2\right] + \left(\lambda v\left(f_\pi^2 - v^2\right) + J\right)\sigma\right)\Bigg). \tag{7}$$

Now we integrate with respect to the σ field. The free σ propagator [4] $\Delta_{\sigma F}$ appears.

$$W[J] = N\exp\Bigg(-\tfrac{i}{2}\int \mathrm{d}^4x'\,\mathrm{d}^4x \; J^a(x')\Delta_{\sigma F}(\mathbf{x}' - \mathbf{x})J_a(x) - \tfrac{i}{2}\int \mathrm{d}^4\bar{x}\,\mathrm{d}^4\bar{x}' \; \left(\lambda v(f_\pi^2 - v^2) + J(x')\right)\Delta_{\pi F}(\mathbf{x}' - \mathbf{x})\left(\lambda v(f_\pi^2 - v^2) + J(x)\right)\Bigg) \tag{8}$$

The **Pressure** will be calculated starting from the renormalized expression

$$\ln Z = \ln W[0] = -\tfrac{\lambda}{4}\left(f_\pi^2 - v^2\right)^2 \beta V + \tfrac{1}{2}\operatorname{tr}\ln \Delta_\beta^\sigma + \tfrac{N-1}{2}\operatorname{tr}\ln\Delta_\beta^\pi; \tag{9}$$

$$\exp \tfrac{1}{2}\operatorname{tr}\ln\Delta_\beta^\sigma = \beta V P_0(T,m) = -V\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\ln\left(1-e^{-\beta\omega}\right), \text{with } \omega^2 = k^2+m^2; \tag{10}$$

$$\therefore P = \frac{T}{V}\ln Z = -\tfrac{\lambda}{4}\left(f_\pi^2 - v^2\right)^2 + VP_0(T,m_\sigma) + V(N-1)P_0(T,m_\pi) \tag{11}$$

We expect that the temperature rise tends to disorder the condensate until it dissapears. In the linear σ model we analyze this phenomena by expanding the free boson gas pressure about zero mass:

$$P_0(T,m) = \frac{\pi^2}{90}T^4 - \frac{m^2T^2}{24} + \frac{m^3 T}{12\pi} + \ldots; \tag{12}$$

$$P(T,v) \approx N\frac{\pi^2}{90}T^4 - \frac{\lambda}{2}v^2\left(f_\pi^2 - \frac{N+2}{12}T^2\right) - \frac{\lambda}{4}\left(v^4 + f_\pi^4 - \frac{N}{6}T^2 f_\pi^2\right). \tag{13}$$

The pressure tends to be a maximum with respect to v:

$$\frac{\mathrm{d}P}{\mathrm{d}v} = \lambda v\left(f_\pi^2 - \frac{N+2}{12}T^2\right) - \lambda v^3 = 0 \qquad\qquad v^2 = f_\pi^2 - \frac{N+2}{12}T^2 \tag{14}$$

The condensate goes to zero at a critical temperature given by $T_c^2 = \frac{12}{N+2}f_\pi^2$. Above this temperature the condensate is zero, because thermal fluctuations are very large. Let $P_{\gtrless}(T)$ be the pressure with $T \gtrless T_C$. The transition is of second-order:

$$P''_<(T_c) = N\frac{2\pi^2}{15}T_c^2 + \lambda\left(\tfrac{N+2}{12}\right)^2 2T^2 + \frac{\lambda N}{12}f_\pi^2 = P''_>(T_c) + \lambda\left(\tfrac{N+2}{12}\right)^2 2T^2. \tag{15}$$

3. Nonlinear σ Model

This model is defined by the Lagrangian

$$\mathscr{L} = \tfrac{1}{2}\left(\partial_\mu \mathbf{\Phi}\right)^2, \text{ with the constraint } f_\pi^2 = \Phi^2(x,t) \tag{16}$$

We begin by representing the field-constraining δ function by an integral.

$$Z = \int \mathscr{D}\Phi\mathscr{D}b'\delta(f_\pi^2 - \Phi^2)\exp\int \mathrm{d}^4x\ (\mathscr{L}) = \int \mathscr{D}\Phi\mathscr{D}b'\exp\int\mathrm{d}^4x\ \left(\mathscr{L} + ib'\left(\Phi^2 - f_\pi^2\right)\right). \tag{17}$$

The Fourier expansions of the fields are

$$\pi_i(\mathbf{x},\tau) = \phi_i(\mathbf{x},\tau) = \sqrt{\frac{\beta}{V}}\sum_{\mathbf{p},n} e^{i(\mathbf{x}\cdot\mathbf{p}+\omega_n\tau)}\tilde{\pi}_i(\mathbf{p},n); \qquad \sigma(\mathbf{x},\tau) = v + \phi_n(\mathbf{x},\tau) \tag{18}$$

$$b'(\mathbf{x},\tau) = i\frac{m^2}{2} + b(\mathbf{x},\tau) = i\frac{m^2}{2} + T\frac{\beta}{V}\sum_{\mathbf{p},n} e^{i(\mathbf{x}\cdot\mathbf{p}+\nu_n\tau)}\tilde{b}(\mathbf{p},n), \tag{19}$$

where we define the abbreviation $\sum_{\mathbf{p},n} \equiv \sum n \int \mathrm{d}^3p(2\pi)^3$. The zero frequency and zero momentum modes have been excluded from the summations. The field Φ must be periodic in imaginary time for the usual reasons but there is no such requirement on b, hence

$\omega_n = 2\pi nT$ and $\nu_n = \pi nT$. An effective action is derived by expanding e^S in powers of b and integrating over the pion and σ fields the term quadratic in b. We get

$$S_E \approx -\frac{1}{2}\sum_{\mathbf{p},n}\left(\omega_j^2+\mathbf{p}^2+m^2\right)\Big[\tilde{\boldsymbol{\pi}}(\mathbf{p},n)\cdot\tilde{\boldsymbol{\pi}}(-\mathbf{p},-n)+\sigma(\mathbf{p},n)\sigma(-\mathbf{p},-n)\Big]$$
$$-\frac{1}{2}\sum_{\mathbf{p},n}\left[\Pi+\frac{2}{N}\frac{v^2}{\omega_n^2+p^2+m^2}\right]\tilde{b}(\mathbf{p},2n)\tilde{b}(-\mathbf{p},-2n)+\tfrac{1}{2}m^2(f_\pi^2-v^2)\beta V \tag{20}$$

where the b one-loop propagator is given by

$$\Pi = \Pi(\mathbf{p},\omega_n,T,m) = T\sum_i\int\frac{\mathrm{d}^3x}{(2\pi)^3}\frac{1}{(\omega_n-\omega_l)^2+(\mathbf{p}-k)^2+m^2}\frac{1}{\omega_l^2+k^2+m^2} \tag{21}$$

This gives rise to the effective pressure of $N-1$ Goldstone bosons with a term coming out of the $(1/N)^2$ expansion in chiral perturbation theory. We have

$$P = (N-1)\frac{\pi^2}{90}T^4 - \tfrac{N+2}{24}m^2T^2+\tfrac{1}{2}m^2(f_\pi^2-v^2)+\tfrac{N}{12\pi}m^3T \tag{22}$$

We maximize in the high temperature phase where $v=0$. This gives the same critical temperature as in the mean field treatment of the linear σ model:

$$\frac{\mathrm{d}P}{\mathrm{d}m}=0 \;\therefore f_\pi^2 = T^2\left[\tfrac{N+2}{12}-\tfrac{N}{4\pi}\tfrac{m}{T}\right]\;\therefore T_C^2=\frac{12}{N+2}f_\pi^2. \tag{23}$$

The transition is also of second order:

$$P''_>(T_C) = N\frac{\pi^2}{90}12T_c^2-\frac{N}{6}\frac{\pi^2}{9}T_C = P''_<(T_C)-\frac{N}{6}\frac{\pi^2}{9}T_C \tag{24}$$

4. Concluding remarks

In the nonlinear σ model, the length of the chiral field is fixed and there can be no chiral-symmetry restoring phase transition. Taking the limit $\lambda\to\infty$ constraints the length of the chiral field to be f_π just as in the nonlinear model. The critical temperature, however, is independent of λ at least in the mean field approximation. So it would seem that the phase transition survives. An expansion in powers of $1/N$ allowed us to calculate the chiral phase transition in the nonlinear σ model without changing the field length. The equations (15) and (24) allow us to see that the transition is of second order, but this subject will be treated in a future paper.

5. References

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