

ISOSPECTRAL POTENTIALS AND DARBOUX TRANSFORMATIONS

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ABSTRACT

It is known that the Darboux transformation (DT) allows us to construct generalized isospectral potentials in ordinary quantum mechanics. A simple mathematical deduction of the DT is presented, including an application for the Hulthén interaction.

Key words - Darboux transform; isospectral potentials; Hulthén interaction

RESUMEN

Se demuestra que la transformación de Darboux (TD) permite construir potenciales isoespectrales generalizados en mecánica cuántica ordinaria. Se presenta una simple deducción matemática de la TD, incluyéndose una aplicación para la interacción de Hulthén.

Palabras Clave.- Transformada de Darboux; potenciales isoespectrales; interacción de Hulthén

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1. Introduction

The one-dimensional stationary case of the Schrödinger equation is given by [1,2]:

$$-\frac{d^2}{dx^2}\psi + u(x)\psi = \lambda\psi \quad (1)$$

which is written in natural units taking $\frac{\hbar^2}{2m} = 1$. The values of λ represent the energy spectrum

allowed for determined boundary conditions corresponding to the standard potential $u(x)$. With the very useful Darboux transformation (DT) [3-6] is possible to generalize any specific standard potential and thus generate new interaction models with the same energy levels. The DT is related to the Sturm-Liouville theory [7-10], and it is easy to see the implicit presence of DT in supersymmetric quantum mechanics [1,2,5,11-17].

We suppose that (1) accepts the particular solution Ψ_l for the eigenvalue λ_l :

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$$-\psi_1'' + u(x)\psi_1 = \lambda_1\psi_1 \tag{2}$$

then we employ ψ_1 as a “seed function” to construct the DT[3-5,18]:

$$\phi(x) = \psi_1' - \sigma_1(x)\psi_1 \quad , \quad \sigma_1 = \frac{d}{dx} \ln \psi_1 \tag{3}$$

therefore (1) adopts the structure:

$$-\frac{d^2}{dx^2}\phi + U(x)\phi = \lambda\phi \tag{4}$$

with the generalized isospectral potential:

$$U(x) = u(x) - 2\frac{d}{dx}\sigma_1 \tag{5}$$

That is, the Schrödinger equation is covariant with respect to DT. Other “seed functions” can generate many DTs and thus a great family of generalized potentials with the same energy spectrum. In section II we show a simple procedure to motivate (3), (4) and (5), that is, we exhibit how the basic expressions of DT are born.

The DT is a mathematical technique that can be interpreted as supersymmetry [1,2,5,11-15,19,20], when applied in quantum mechanics. In [21-27] Riccati equation [7,28-30] is employed to construct generalized potentials isospectral to the free particle, harmonic oscillator, hydrogenic, Morse [1,13,14,31-36] and Hulthén [1,37-41] interactions. It is possible to obtain all these generalized potentials via an adequate DT, which shows the power of the Darboux’s procedure. As an example, in section III we apply the DT to Hulthén model, as an alternative method to the Riccati equation, in the search of new quantum mechanical potentials for a given spectrum.

2. Darboux transformation

If in (1) we introduce the new dependent variable $y(x) = \psi(x)/\theta(x)$, where $\theta(x)$ is an arbitrary function for the time being, then this equation takes the form:

$$y'' + 2\frac{\theta'}{\theta}y' + (\lambda - \lambda_1 + \frac{\theta''}{\theta} - \frac{\psi_1''}{\psi_1})y = 0 \tag{6}$$

using $u = \lambda_1 + \psi_1''/\psi_1$ from (2). Therefore, it is possible to choose $\theta = \psi_1$, yielding to:

$$y = \frac{\psi}{\psi_1} \tag{7}$$

reducing (6) to:

$$y'' + 2\frac{\psi_1'}{\psi_1}y' + (\lambda - \lambda_1)y = 0 \tag{8}$$

Now we apply $\frac{d}{dx}$ to (8) and introduce the notation:

$$\eta(x) = \frac{d}{dx} y(x) , \quad \sigma_1 = \frac{\psi_1'}{\psi_1} \tag{9}$$

obtaining the equation:

$$\eta'' + 2\sigma_1\eta' + (\lambda - \lambda_1 + 2\sigma_1')\eta = 0 \tag{10}$$

Finally, in (10) we make a transformation similar to (7):

$$\eta = \frac{\Phi}{\psi_1} \tag{11}$$

then these equation adopts the structure of (4) with the generalized isospectral potential $U(x) = \sigma_1'^2 - \sigma_1' + \lambda_1 = u - 2\sigma_1'$, accordingly with (5). Besides, from (7), (9) and (11) we

have that $\phi = \psi_1\eta = \psi_1 y' = \psi_1 \frac{d}{dx} (\psi / \psi_1)$, which reproduces (3) q.e.d.

In the literature on DT there is not an explicit motivation for these important transformations of mathematical physics. Thus, this section was dedicated to a simple demonstration of the basic expressions of DT.

3. Generalized Hulthén Potentials

The Hulthén potential [37] is a useful interaction model that has been used extensively [42] in different areas of Physics, including nuclear [43] and atomic physics [44], due to the fact that it yields to closed analytic solutions for the s waves [39,45,46]. It is given by [27,39]:

$$u(r) = -\frac{V_o}{e^{Ar} - 1} \tag{12}$$

where the screening parameter A and V_o are positive constants such that $V_o > A^2$. This potential is a special case of the Eckart potential [47]. It is clear that [1,2] the Schrödinger equation for the radial wave function $R(r)$ takes the form (1) with $R = \frac{1}{r}\psi$ for $l = 0$.

Here we shall take the example discussed in [48] as a sample of the usefulness of DT. According with equation (3) the DT depends on the function ψ_1 selected, so we will show two options verifying (2):

a) ψ_1 is the usual wave function [39,45] for the ground state associated to (12):

$$\psi_1(r) = (1 - e^{-Ar})e^{-kr} \quad \lambda_1 = -k^2 \tag{13}$$

where $k = \frac{1}{2A}(V_0 - A^2) > 0$

Employing (13), the relations (3) and (5) lead to the generalized potential of Hulthén (Darboux potential):

$$U_m = -\frac{V_0}{e^{Ar} - 1} + \frac{2A^2 e^{Ar}}{(e^{Ar} - 1)^2} \tag{14}$$

equivalent to (36) of [27] and (7) of [49], which is isospectral to (12). U_m has the structure of the approximate Hulthén effective potential introduced by Greene-Aldrich [50] in their method to generate pseudo-Hulthén wave functions for states with non-zero angular momentum.

b) Another possibility is to utilize:

$$\psi_1(r) = \frac{1}{b}(1 - e^{-Ar})e^{-kr} \left[\gamma + b \int \frac{e^{2kr} dr}{(1 - e^{-Ar})^2} \right] \tag{15}$$

satisfying (2) with $\lambda_1 = -k^2$, being γ and b arbitrary constants.

Expressions (3), (5) and (15) imply the following generalized Hulthén interaction:

$$U_g = U_m + \frac{2b}{\rho} \left[\frac{b}{\rho} - 2 \left(k - \frac{A}{e^{Ar} - 1} \right) \right] \tag{16}$$

with $\rho = b(1 - e^{-Ar})e^{-kr}\psi_1$, corresponding to (11) of [27] obtained via Riccati equation.

Finally, we conclude remarking that the Darboux procedure used to generalize a standard potential is a straightforward method, which is far simpler than equivalent approaches employed to find new families of isospectral known potentials [49,51,52]. In particular, with the DT we may find useful potentials in Molecular Physics, for example, the approximate Hulthén effective interaction introduced by Greene-Aldrich [50], to see (14). Besides, the methodology here presented is applicable the study of the principal equations of Mathematical Physics (Hermite, Legendre, Laguerre, Chebyshev, etc.), if first they are transformed to Schrödinger type equations.

References

- [1] O.L de Lange and R.E. Raab, Operators methods in quantum mechanics, Clarendon Press, Oxford (1991)
- [2] F. Schwabl, Quantum mechanics, Springer-Verlag, Berlin (1992)
- [3] G. Darboux, Compt. Rend. Acad. Sc. (Paris) 94, 1456 (1882)
- [4] A. Khare and U. Sukhatme, J. Phys. A: Math. Gen. 22, 2847 (1989)
- [5] V.B.Matveev and M-A. Salle, Darboux transformations and solitons, Springer-Verlag, Berlin (1991)
- [6] J. Morales, J.J. Peña and J. López- Bonilla, J. Math. Phys. 42, 966 (2001)
- [7] C. Lanczos, Linear differential operators, D. van Nostrand, London (1961)
- [8] H. Hochstadt, The functions of mathematical physics, Dover NY (1986)
- [9] J.B. Seaborn, Hypergeometric functions and their applications, Springer-Verlag, Berlin (1991)
- [10] Z. Ahsan, Differential equations and their applications, Prentice Hall, India (2000)
- [11] A.A. Andrianov, N.V. Borisov and M.J. Ioffe, Theor. Math. Fiz. 61(1), 17 (1984) and 61(2), 183 (1984)
- [12] A.A. Andrianov, N.V. Borisov and M.J. Ioffe, Phys. Lett. B181, 141(1986)
- [13] R.W. Haymaker and A.R.P.Rau, Am. J.Phys. 54, 928 (1986)
- [14] F.Cooper, A.Khare and U.Sukhatme, Phys. Rep. 251, 267 (1995)
- [15] H.R. Jauslin, Helv. Phys. Acta 61,901 (1988)
- [16] B. Bagchi, Supersymmetry in quantum and classical mechanics, Chapman & Hall, Florida – USA (2000)
- [17] F. Cooper, A. Khare and U.Sukhatme, Supersymmetry in quantum mechanics, World Scientific Pub. Singapore (2001)
- [18] M.Crum, Quat. J. Math. 6, 121 (1955)
- [19] D. Yi – Bing Ding, J. Phys. A: Math. Gen. 20, 6293 (1987)
- [20] H. Rosu and J.R. Guzmán, Nuovo Cim. B112, 941 (1997)
- [21] B. Mielnik, J. Math. Phys. 25, 3387(1984)
- [22] D.J. Fernández, Lett. Math. Phys. 8, 337 (1984)
- [23] Zhu Dongpei, J. Phys. A: Math. Gen. 20, 4331 (1987)
- [24] J. Morales and J.J. Peña, J. Math. Phys. 40, 5555 (1999)
- [25] J. Morales, J. J. Peña, V. Gaftoi and G. Ovando, Int. J. Quantum Chem. 71, 465 (1999)
- [26] J. Morales and J. J. Peña, J. Mol. Struct. (Techoem) 493, 43 (1999)
- [27] J. Morales, J. J. Peña, and J. D. Morales – Guzmán, Theor. Chem. Acc. 104, 179 (2000)
- [28] H. T. Davis, Introduction to nonlinear differential and integral equations, Dover NY (1962)
- [29] D. J. Struik, Lectures on classical differential geometry, Douver NY (1988)
- [30] S. Bittanti, A. J. Laub and J.C. Willems, The Riccati equation, Springer – Verlag (1991)
- [31] P. M. Morse, Phys. Rev. 34, 57 (1929)
- [32] S. Y. Lee, Am. J. Phys. 53, 753 (1985)
- [33] M. Berrondo, A. Palma and J. López – Bonilla, Int. J. Quantum Chem. 31, 243 (1987)
- [34] A. Matsumoto and K. Iwamoto, J. Quant. Spectrosc. Radiat. Transfer 50, 103 (1993)
- [35] J. López – Bonilla, D. Navarrete, H. N. Núñez – Yépez and A. L. Salas – Brito, Int. J. Quantum Chem. 62, 177 (1997)
- [36] J. H. Caltenco, J. López – Bonilla and R. Peña – Rivero, J. Sci. Res. 50, 125 (2000)
- [37] L. Hulthén, Ark. Mat. Astron. Fys. 28, 5 (1942)
- [38] P. Matthys and H. De Meyer, Phys. Rev. A38, 1168 (1988)
- [39] O. L. de Lange, Am. J. Phys. 59, 151 (1991)
- [40] R. L. Hall, J. Phys. A: Math. Gen. 25, 1373 (1992)
- [41] H.S. Green, Matrix mechanics, Noordhoff – Groningen (1965)
- [42] Y.P.Varshni, Phys. Rev. A41, 4682 (1990)
- [43] R.L. Hall, Phys. Rev. A32, 14 (1985)

- [44] U. Myhrman, J. Phys. A: Math. Gen. 16, 263 (1983)
- [45] S. Flugge, Practical quantum mechanics, Springer – Verlag (1974)
- [46] C.S. Lam and Y. P. Varshni, Phys. Rev. A4, 1875 (1971)
- [47] C. Eckart, Phys. Rev. 35, 1303 (1930)
- [48] J. López-Bonilla, J. Morales and G. Ovando, Apeiron 9, 20(2002)
- [49] B. Gönül, O. Özer, Y. Cancelik and M Kocak, Phys. Lett. A275, 238 (2000)
- [50] R. L. Greene and C. Aldrich, Phys. Rev. A14, 2363(1976)
- [51] J. O. Rosas – Ortiz, J. Phys. A: Math. Gen. 31, L507 (1998)
- [52] J. Morales, J. J. Peña and J. López – Bonilla, J. Mol. Struct. (Theochem) 621, 19 (2003)