ELECTRIC FIELD PRODUCED BY THE ILLUMINATION OF A PLANE OBJECT WITH A SPHERICAL WAVE: THE FRACTIONAL FOURIER TRANSFORM APPROACH

César O. Torres[1,2], Yezid Torres[1] and Pierre Pellat-Finet[3♦]
[1]*Grupo de Optica y Tratamiento de Señales Universidad Industrial de Santander*
A.A. 678. Bucaramanga-Colombia, e-mail: ytorres@uis,edu.co
[2]*Laboratorio de Optica e Informática, LOI, Universidad Popular del Cesar*
A.A. 590. Valledupar-Colombia
Currently at the Grupo de Optica y Tratamiento de Señales, UIS.
[3]*U.E.F. de Sciencies Université de Bretagne Sud, Lorient, France*
(Recibido 15 Jun. 2005; Aceptado 15 Ago. 2005; Publicado 23 Dic. 2005)

RESUMEN

Aquí se demuestra que la distribución de amplitud luminosa sobre un detector esférico puede escribirse en función de una transformación de Fourier de orden fraccional, cuando un objeto plano es iluminado adecuadamente con una onda esférica. Esta propuesta permite establecer una teoría general de la propagación de una onda en el espacio libre. Este resultado provee una alternativa de la ley de propagación y permite usar la transformada de Fourier de orden fraccional como una herramienta para el análisis y descripción de una clase más general de sistemas ópticos.

Palabras Clave: Transformada de Fourier de orden fraccional, Difracción, Óptica de Fourier.

ABSTRACT

In this work it is shown that the amplitude distribution of light on a spherical detector can be written in terms of the fractional Fourier transform when a planar object is illuminated with an adequate spherical wave. Here, the approach proposed obtains a general theory of the propagation of wave in free space. This result provides an alternative statement of the law of propagation and allows us to use the fractional Fourier transform as a tool for analyzing and describing a rather general class of optical system.

Key words: Fractional Fourier transform, Diffraction, Fourier optics.

1. Introduction

In figure 1, $\vec{r}$ is the vector of the corresponding coordinates on plane A and $\vec{r'}$ is the vector of coordinates of the observation plane P. Let $u_A\left(\vec{r}\right)$ be the field complex amplitude at

♦ e-mail: pierre.pellat-finet@univ-ubs.fr

point $\vec{r}$ of diffracting screen A and $u_P(\vec{r})$ the field complex amplitude at point $\vec{r'}$ of detector P.

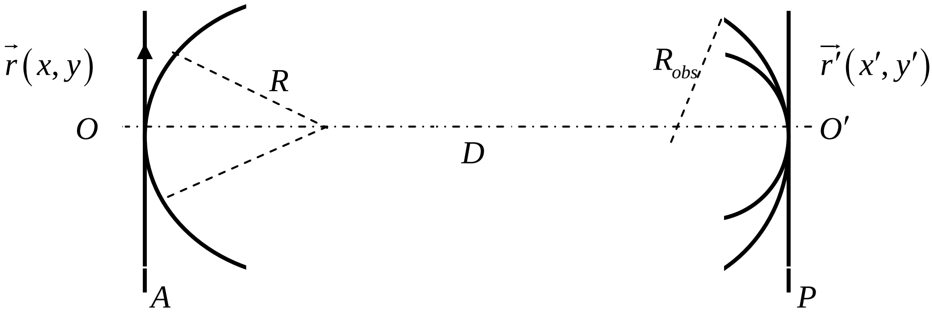

Figure 1. Fresnel diffraction of plane screen A observed on screen P when is illuminated by a spherical wave of radius R.

When the planar object is illuminated by a plane wave, the propagation of A to P is related to Fresnel diffraction[1] as:

$$u\left(\vec{r'}\right)=\frac{i}{\lambda D}\exp\left[\frac{-i\pi r'^2}{\lambda D}\right]\int_{R^2}\exp\left[\frac{-i\pi r^2}{\lambda D}\right]\exp\left[\frac{2i\pi}{\lambda D}\vec{r}\cdot\vec{r'}\right]u\left(\vec{r}\right)d\vec{r} \qquad (1)$$

Where λ and D are respectively the wavelength and the distance between the screen A and the screen P.

Notice that in Eq. (1) a factor $\exp\left[\frac{-2i\pi D}{\lambda}\right]$ has been neglected. This factor is related to the propagation time from A to P; it can be omitted if the time origin on P is shifted with respect to the time origin on A.

2. Convergent Spherical Wave Illumination and Huygens principia

Using the convergent spherical wave of radius R for illuminating the plane object located in A, the field complex amplitude on the plane detector P becomes:

$$u_P\left(\vec{r'}\right)=\frac{i}{\lambda D}\exp\left[-\frac{i\pi r'^2}{\lambda D}\right]\int_{R^2}\exp\left[-\frac{i\pi r^2}{\lambda D\left(\frac{R}{R-D}\right)}\right]\exp\left[\frac{2i\pi}{\lambda D}\vec{r}\cdot\vec{r'}\right]u_A\left(\vec{r}\right)d\vec{r} \qquad (2)$$

Selecting appropriate scaled variables[2] :

$$\vec{\rho} = \sqrt{\frac{2\pi\left(1-\frac{\mathrm{D}}{\mathrm{R}}\right)\tan\alpha}{\lambda\mathrm{D}}}\,\vec{r} \text{ and, } \vec{\sigma} = \sqrt{\frac{2\pi\sin\alpha\cos\alpha}{\lambda\mathrm{D}\left(1-\frac{\mathrm{D}}{\mathrm{R}}\right)}}\,\vec{r}'$$

and using the definition of the fractional Fourier transform[3,4], we conclude that the field complex amplitude over the plane detector $u_P\left(\vec{r}'\right)$ is proportional to the fractional Fourier transform of order α of the field complex amplitude of the planar object $u_A\left(\vec{r}\right)$; then:

$$u_P\left(\vec{r}'\right) = \frac{i}{\lambda D}\sqrt{2\pi\sin\alpha}\,\exp\left(-i\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\right)\exp\left[\frac{-i\pi r'^2}{\lambda D\left(\frac{D-R}{D-R\sin^2\alpha}\right)}\right]TFF^{\alpha}\left\{u_A\left(\vec{r}\right)\right\} \qquad (3)$$

If $R = D$, illumination of plane object with a convergent wave, the last expression becomes a standard Fourier transform over spherical concave receiver of radius D[1] and variables change is not necessary.

The factor $\exp\left[-\frac{i\pi r'^2}{\lambda D\left(\frac{D-R}{D-R\sin^2\alpha}\right)}\right]$ in equation (3) is a quadratic phase factor representing a quadratic approximation of a spherical wave diverging from a luminous point at distance $R_{obs} = -D\left[\frac{D-R}{D-R\sin^2\alpha}\right]$ from P.

The radius of curvature is taken from vertex to center. It can be positive or negative according to the convexity of the spherical segment.

This quadratic factor is compensated if the electric field is observed on spherical surface of radius Robs and not on plane P; in this way the field amplitude on Robs is exactly the fractional Fourier transform of order α of the field amplitude on plane A. In other words, the complex amplitude of the electric field on the spherical detector of radius Robs is proportional to the exactly fractional Fourier transform of order α of the planar object.

An interesting property of fractional order Fourier transforms is their continuity with respect to their orders[3,4]. If fractional order α is observed on spherical receiver of radius R^{α}_{obs}, the planar object is set at distance D and illuminated by a spherical wave of radius R and the fractional order β observed on a spherical receiver of radius R^{β}_{obs}, the same planar object placed at distance D' and illuminated by the same spherical wave, then TFF^{β} tends to TFF^{α} as D' tends to D. In the same sense, if the fractional order γ is observed on spherical receiver of radius R^{γ}_{obs}

when the planar object placed at the same distance D illuminated by spherical wave of radius R', then TFF$^{\gamma}$ tends to TFF$^{\alpha}$ as R' tends to R.

For D fixed and finite, the radius R of the lighting spherical wave can be changed and the corresponding radius Robs of the spherical receiver must obeys the condition above indicated for observing the desired fractional order.

The fractional Fourier transforms of different orders are defined as the amplitude field distributions as they propagate along the medium between the input and the Fourier planes; using either operation, the fractional Fourier transform or Fresnel diffraction, permits a continuos transition from the object domain to be accomplished. Accordingly, the order varies from $\alpha=0$ (pure spatial information) to $\alpha=\pi/2$ (pure spectral information).

We now discuss these consequences from three perspectives.

3. Case 1. D → 0

The condition $\sin^2\alpha = \left(\frac{D-R}{R\,R_{obs}} + \frac{1}{R}\right)D$ for any R, then the order is $\alpha \to 0$.

The observed fractional Fourier transform over any Robs corresponds at the identity operator.

4. Case 2. D is a finite distance different from zero.

Situation A:

If $R = D$, from equation (2) $R_{obs} = -D$ and from condition, α tends to $\pi/2$. The radius R of the lighting wave and the spherical receiver surface are centered and separated by the distance D.

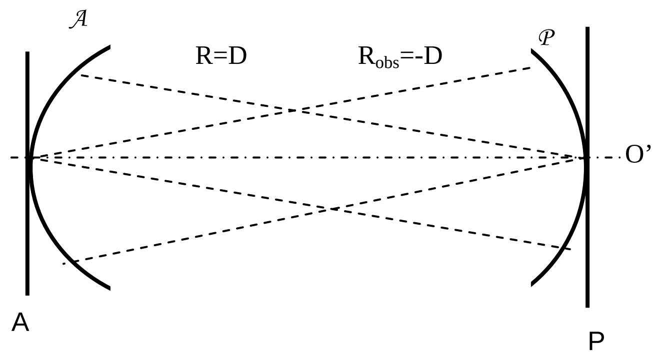

Figure 2. Fraunhofer diffraction: The field complex amplitude on 𝒫 is the Fourier transform of the field amplitude on 𝒜.

This corresponds to a Fraunhofer diffraction pattern observed near the center of curvature of a spherical receiver. Equation (2) involves a Fourier transform and for this reason, sphere 𝒫 is called the "Fourier sphere of 𝒜 "[5].

The issue for different R, D and Robs (for instance, resonators) has been analyzed by H.M.Ozaktas and D.Mendlovic in reference [6], and this result is in full agreement with them.

Situation B:

If $R_{obs} \to \infty$, we want to make the observation on the plane receiver (very interesting for the experimental reasons), then the radius of the spherical wave illumination must be: $R = \dfrac{D}{\sin^2 \alpha}$; α ≠ 0, π/2. This means that the fractional order Fourier transform of order α can be observed at a distance D on the plane P if the illumination wave complies with the curvature given by this condition (for real α, only if $R > D$). If the convergent spherical wave illumination radius change, for example increase, the fractional Fourier order α decrease for distance D fixed. And for a spherical wave illumination of radius R fixed, $\sin\alpha$ increase with distance following $\sqrt{D}$.

Situation C:

If $R \to \infty$, illumination be made with a plane wave, then the radius of the spherical receiver is $R_{obs} = -\dfrac{D}{\sin^2 \alpha}$; α ≠ 0, π/2. At distance D there is an infinite number of spheres where a fractional order Fourier transform can be observed[2]. Let S be a sphere with the radius Robs; the corresponding fractional Fourier transform has an order such that last condition is accomplished, which is possible for real α, only if $R_{obs} \le -D$. If the concave surface detector radius changes, for instance increase, the order α of fractional Fourier transform decrease for D fixed. And for a surface detector radius Robs fixed, $\sin\alpha$ increases with distance following the same law for last situation.

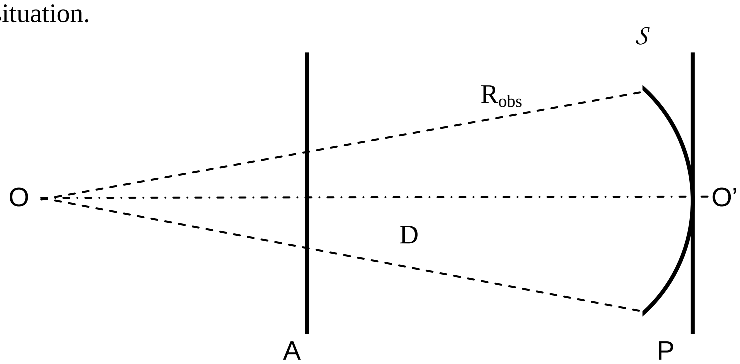

Figure 3. The field amplitude on $\mathcal{S}$ is obtained from the field amplitude on A with a fractional Fourier transform of order α

Situation D:

The issue of observing the fractional Fourier transform in free space for D finite, when the size of the object is scaled by $\cos\alpha$ [7], can be deduced from equation (3). So, when we want to observe the free-space Fresnel diffraction pattern produced by an object scaled with a reduction factor, for only low orders, the intensity distribution on the plane of detection is given by the fractional Fourier transform of the object. When the object pattern is scaled with an enlarging factor, the mathematical expression for the intensity distribution at the observation plane is also the object's fractional Fourier transform. These results are in accordance with the experimental report of H.Jianwen et al. [8].

5. Case 3. D $\rightarrow \infty$

The condition $0 \le \left(\frac{D-R}{R R_{obs}} + \frac{1}{R} \right) D \le 1$ means that R and $R_{obs} \rightarrow \infty$, then α tends to $\pi/2$. This configuration corresponding to lighting with a plane wave, observing on the plane receiver and the distance between the two planes is infinity. The diffracted field at infinite tends to the angular spectrum of $u_A\left(\vec{r}\right)$ [2], which is expressed with a standard Fourier transform; physically, the diffraction phenomenon becomes a Fraunhofer diffraction[1].

6. Conclusions

In this paper we have studied diffraction produced by lighting the plane diffracting screen with a spherical wave, in terms of the fractional Fourier transform; it was found that in free space the field complex amplitude on the plane receiver is proportional to the fractional Fourier transform of the field complex amplitude of the plane emitter multiplicated by a phase factor.

The phase factor was investigated from various possibilities for the parameters involved; obtaining a general mathematical expression for the physical descriptions of the propagation of the electric field between surfaces in free space when lighting with any spherical wave.

The fractional Fourier transform order change with distance in free space; if distance is zero, the identity operator is observed over any spherical detector (pure spatial information); for finite distances any one order ($0< \alpha \le \pi/2$) can be observed over a appropriate spherical detector using a suitable spherical wave illumination. Finally, for infinity distance, order is $\pi/2$ (pure spectral information).

References

[1] W. Goodman, "Introduction to Fourier Optics", McGraw-Hill, New York, Chap. 4, p. 60, (1968).
[2] P. Pellat-Finet, "Fresnel diffraction and the fractional-order Fourier transform", Opt. Let. 19, 1388-1390 (1994).
[3] V. Namias, "The fractional Fourier transform and its applications to quantum mechanics", J. Inst. Math. Appl. 25, 241-265 (1980).
[4] C. McBride and F. H. Kerr, "On Namia´s, fractional Fourier transforms", IMA J. Appl. Math. 39, 159-175 (1987).
[5] P. Pellat-Finet and G. Bonnet, "Fractional order Fourier transform and Fourier optics", Opt. Com. 111, 141-154 (1994).
[6] H.M. Ozacktas and D. Mendlovic, "Fractional Fourier optics", J. Opt. Soc. Am. A 12, 743-751 (1995).

[7] C.O. Torres, Y. Torres and P. Pellat-Finet,. “The free space Fresnel diffraction and the fractional Fourier transform”, Edición electrónica Academia Colombiana de Ciencias Exactas, Físicas y Naturales, ISBN 958-9205-32-1. Comunicación 090, Santafé de Bogotá, Octubre, (1998).

[8] H. Jianwen, L. Liren and L. Guoqiang, “Observing the fractional Fourier transform by free-space Fresnel diffraction”, Applied Optics 36, 512-513 (1997).